AN INVESTIGATION OF SOME CONJECTURES ABOUT THE HAZARD RATE OF WARM STANDBY SYSTEMS

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THESIS

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ABSTRACT

Two propositions concerning the comparative behavior of the hazard rate functions for warm, cold, and hot standby systems are expressed and then examined. The standby systems under study involve two components with constant failure rates. Emphasis is placed on establishing a lower bound on the hazard rate function for a standby system and on developing an optimal employment policy for the use of components in a standby system.

A proof is offered to show that the hazard rate function for a cold standby system is a lower bound on the hazard rate function of either a warm or hot standby system. Evidence is then given to support the existence of an optimal employment policy for components in a standby system, but a complete proof is not presented.

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* Figures 3.1, 3.2, 4.1, and 4.2 are reproduced from Scroggins (Ref. 1) with the permission of the author.

I. INTRODUCTION

Standby systems, involving one functioning component backed up by one spare component, have been studied by many theoreticians. For the standby system, the survivability of the system depends on either one or the other of the two components operating effectively.

If the spare component suffers no deterioration and is not active until the working component fails, the spare component is considered to be in the cold standby mode. If both components are subject to equal usage and deterioration, then the "spare" component is said to be in the hot standby mode. However, it is more plausible that a spare component should be in neither the hot nor cold standby mode, but rather in what is considered the warm standby mode. In this mode, the spare component does not suffer wearout by operation as in the hot standby mode, but rather it suffers from a similar environment of heat, vibration, etc. This mode is intuitively more appealing as even a spare on a shelf will deteriorate to some degree.

Two components in a standby system, where the components have exponential survival distributions, have been previously studied in depth for the cases of either hot or cold standby modes. These two cases are, however, special cases of the warm standby mode. At present, the warm standby system has not been looked at in as much detail. Scroggins (Ref. 1) graphed the hazard rate function for the warm standby system. These graphs led to some interesting conjectures.

The concern in this thesis is to further explore two of these conjectures. The first deals with establishing a lower bound for the

hazard rate function of a warm standby system. It is obvious that the survival function of the cold standby system is a lower bound for the survival function of the warm standby system. It is not, however, so obvious that the hazard rate function of the cold standby system is a lower bound for the hazard rate function of the warm standby system. The second conjecture of interest is about a question of greater applicability. That is, given two components with known failure rates, how should they be employed in the system? Stated another way, is there a component deployment policy which if followed would stochastically maximize the system life? Investigation of this conjecture, although not complete, suggests that such an optimum policy does exist.

II. THE HAZARD RATE FUNCTION

A. THE SURVIVAL FUNCTION

The survival function is defined as $\bar{F}(t) = P(T>t)$, $t \geq 0$; where $T \geq 0$ is the random time to failure of the system. In words this says, $\bar{F}(t)$ is the probability that the system lives longer than time t.

The warm standby system with constant failure rate components in the system is illustrated in Figure 2.1.

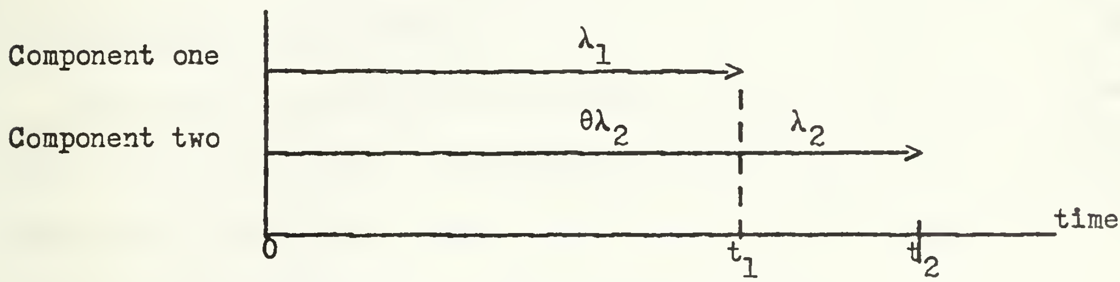

FIGURE 2.1. LIFE DISTRIBUTION OF T

Component one operates until failure occurs, at which time component two takes over if it is still capable of functioning properly. Component one has failure rate λ_1. Component two has failure rate $\theta\lambda_2$, where $0 \leq \theta \leq 1$, until component one fails and is replaced by component two. After it is activated, component two operates with failure rate λ_2.

This is the general form for the warm standby case as the spare does suffer some deterioration with time, but it is not fully affected until it is placed on-line in the system. If $\theta = 0$, the system is a cold standby system. If $\theta = 1$, the system is a hot standby system.

The warm standby system's survival function is

$$\bar{F}(t)=e^{-\lambda_1 t}+e^{-\lambda_2 t}\int_0^t \lambda_1 e^{-(\lambda_1-(1-\theta)\lambda_2)s}\,ds,\ t\geq 0 .$$

From this equation it can be shown that for $0 \leq \theta \leq 1$,

$$(2.1) \quad \overline{F}(t) = \frac{(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-(\lambda_1+\theta\lambda_2)t}}{\lambda_1-(1-\theta)\lambda_2}, \quad t \geq 0,\ \lambda_1 \neq (1-\theta)\lambda_2 .$$

There is a special form of $\overline{F}(t)$ for the case when $\lambda_1 = (1-\theta)\lambda_2$, but this will not be needed in this thesis. The survival function is continuous in θ, λ_1, and λ_2, and no generality is lost by dismissing this special case.

B. THE HAZARD RATE FUNCTION

The hazard rate function is defined in terms of the survival function and its derivative. The hazard rate function is $r(t) = f(t)/\overline{F}(t)$, where $f(t) = -d\overline{F}(t)/dt$. Again, ignoring the special case, for all $0 \leq \theta \leq 1$ this yields

$$(2.2) \quad f(t) = \frac{\lambda_1(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1(\lambda_1+\theta\lambda_2)e^{-(\lambda_1+\theta\lambda_2)t}}{\lambda_1-(1-\theta)\lambda_2}, \quad t \geq 0,$$

which gives

$$(2.3) \quad r(t) = \frac{\lambda_1(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1(\lambda_1+\theta\lambda_2)e^{-(\lambda_1+\theta\lambda_2)t}}{(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-(\lambda_1+\theta\lambda_2)t}}, \quad t \geq 0.$$

C. NORMALIZATION OF THE HAZARD RATE FUNCTION

For ease in working with the hazard rate function, it is advantageous to assume that $\lambda_1 + \lambda_2 = 1$. The scale for time to failure can be chosen so that no generality is lost by this assumption.

The hazard rate will be assumed to be in normalized form ($\lambda_1 + \lambda_2 = 1$) for the remainder of this thesis. The hazard rate

function presented here is derived, and the details of the normalization are discussed in greater detail, in Scroggins (Ref. 1).

III. LOWER BOUND ON THE HAZARD RATE OF THE STANDBY SYSTEM

A. PROPOSITION OF THE EXISTENCE OF A LOWER BOUND

Scroggins (Ref. 1) advanced the conjecture, exemplified by Figures 3.1 and 3.2, that for any given components, the hazard rate for $\theta = 0$ is less than or equal to the hazard rate for $0 \leq \theta \leq 1$. This indicates that the cold standby system's hazard rate is a lower bound on the hazard rate of either the warm standby system or the hot standby system. Such a conclusion would agree with intuition, as a cold standby spare component suffers no deterioration until it is placed on-line in the system.

The above conjecture can be formalized by the following proposition, where $r_0(t)$ denotes the hazard rate of the warm standby system in which $\theta = 0$.

> Given two components, with normalized constant failure rates λ_1 and λ_2, then
>
> $$r_0(t) \leq r_\theta(t),$$
>
> for all $0 \leq \theta \leq 1$ and $t \geq 0$.

B. PROOF

As before, the continuity of the hazard rate allows the special case where $\lambda_1 = (1 - \theta)\lambda_2$ to be ignored. For the general case, where $\lambda_1 \neq (1 - \theta)\lambda_2$, the hazard rate function from Equation 2.3 is

$$r_\theta(t) = \frac{\lambda_1(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1(\lambda_1+\theta\lambda_2)e^{-(\lambda_1+\theta\lambda_2)t}}{(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-(\lambda_1+\theta\lambda_2)t}}.$$

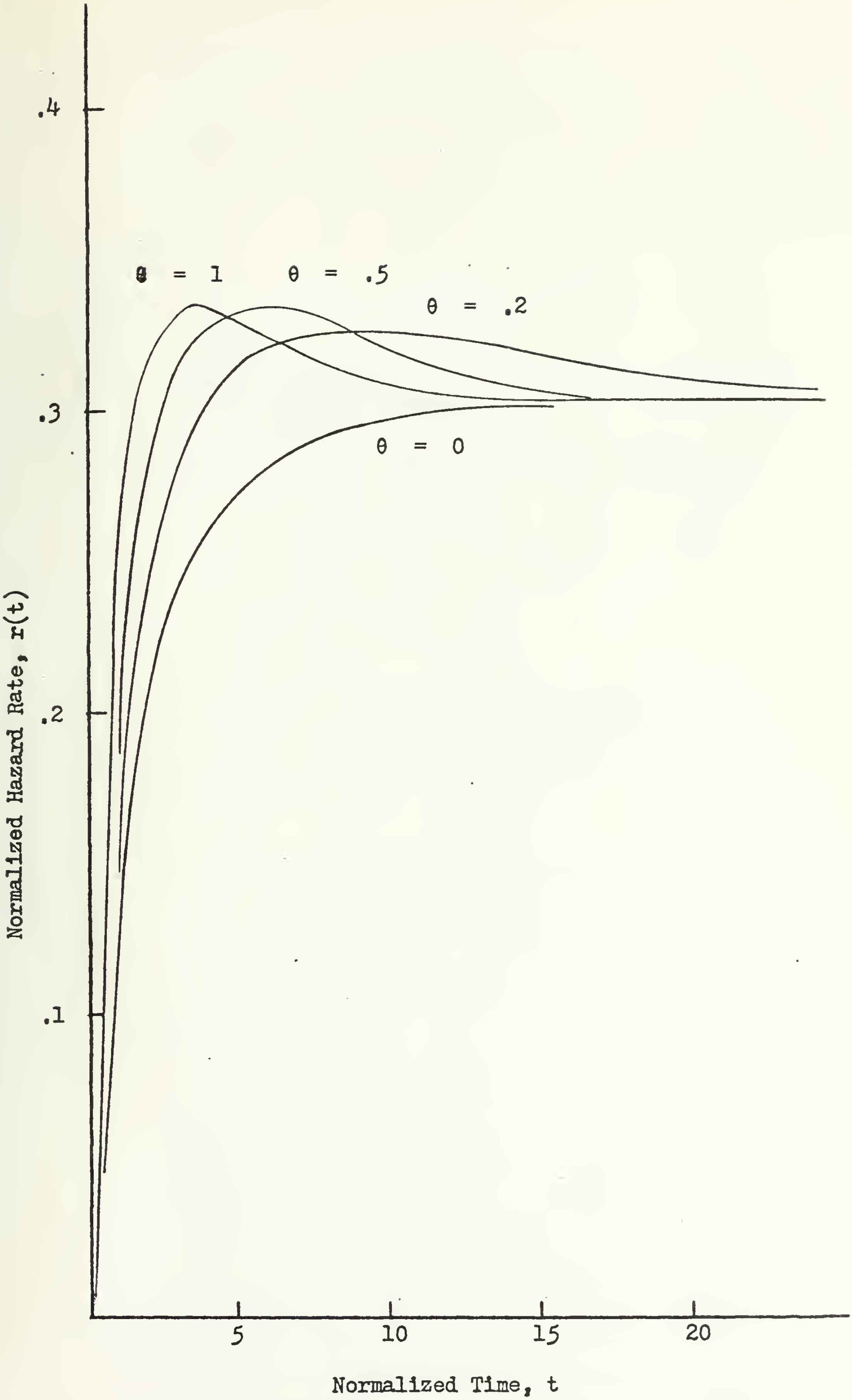

FIGURE 3.1. $\lambda_1 = .3$, VARYING θ

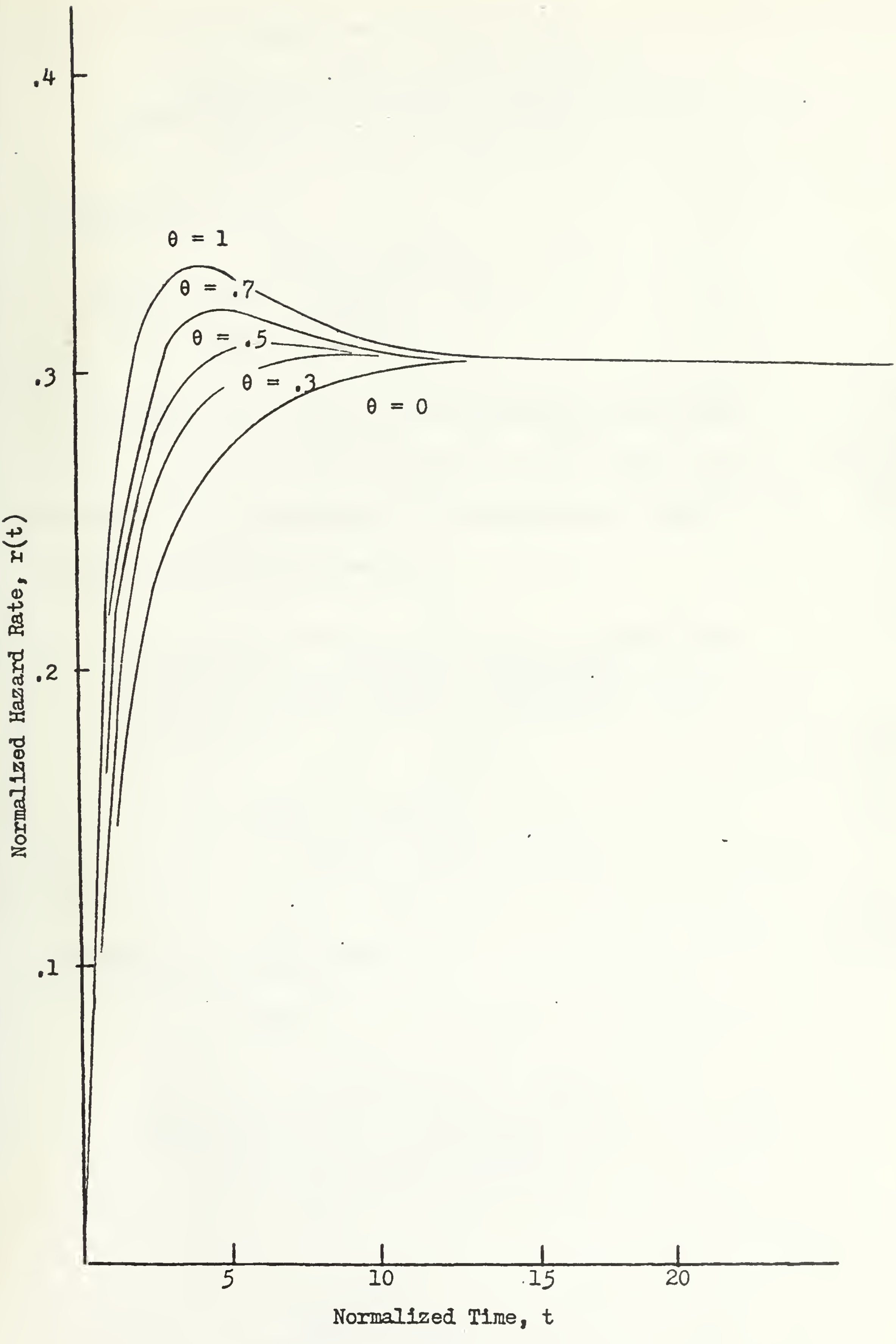

FIGURE 3.2. $\lambda_1 = .7$, VARYING θ

When $\theta = 0$, the hazard rate function becomes

$$r_0(t) = \frac{\lambda_1(\lambda_1-\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1^2 e^{-\lambda_1 t}}{(\lambda_1-\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-\lambda_1 t}},$$

which can be rewritten

$$r_0(t) = \frac{\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1\lambda_2 e^{-\lambda_1 t}}{\lambda_1 e^{-\lambda_2 t}-\lambda_2 e^{-\lambda_1 t}}.$$

To prove $r_0(t) \leq r_\theta(t)$, it is sufficient to show that $r_\theta(t) - r_0(t) \geq 0$, for all $0 \leq \theta \leq 1$, $t \geq 0$, and $\lambda_1 + \lambda_2 = 1$. Substituting in the expressions for the hazard rate functions, the inequality to be proved becomes

$$\frac{\lambda_1(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1(\lambda_1+\theta\lambda_2)e^{-(\lambda_1+\theta\lambda_2)t}}{(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-(\lambda_1+\theta\lambda_2)t}}$$

$$+\ \frac{\lambda_1\lambda_2 e^{-\lambda_1 t}-\lambda_1\lambda_2 e^{-\lambda_2 t}}{\lambda_1 e^{-\lambda_2 t}-\lambda_2 e^{-\lambda_1 t}} \geq 0.$$

Multiplying by $\dfrac{\lambda_1 e^{\lambda_1 t}}{\lambda_1\lambda_2 e^{\lambda_1 t}}$ gives

$$\frac{(\frac{\lambda_1}{\lambda_2}-(1-\theta))+e^{-(\lambda_2-\lambda_1)t}-(\frac{\lambda_1}{\lambda_2}+\theta)e^{-\theta\lambda_2 t}}{(1-(1-\theta)\frac{\lambda_2}{\lambda_1})+e^{-(\lambda_2-\lambda_1)t}-e^{-\theta\lambda_2 t}}$$

$$+\ \frac{1-e^{-(\lambda_2-\lambda_1)t}}{e^{-(\lambda_2-\lambda_1)t}-\frac{\lambda_2}{\lambda_1}} \geq 0.$$

Rearranging terms, this becomes

$$\frac{(e^{-(\lambda_2-\lambda_1)t}-1) + \frac{\lambda_1}{\lambda_2}(1-e^{-\theta\lambda_2 t}) + \theta(1-e^{-\theta\lambda_2 t})}{(e^{-(\lambda_2-\lambda_1)t}-\frac{\lambda_2}{\lambda_1}) + (1+\frac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t})}$$

$$+ \frac{(1-e^{-(\lambda_2-\lambda_1)t})}{(e^{-(\lambda_2-\lambda_1)t}-\frac{\lambda_2}{\lambda_1})} \geq 0.$$

Demonstrating this inequality as true is equivalent to showing that

$$(3.1)\quad \Big[(e^{-(\lambda_2-\lambda_1)t}-\tfrac{\lambda_2}{\lambda_1})[(e^{-(\lambda_2-\lambda_1)t}-1)+\tfrac{\lambda_1}{\lambda_2}(1-e^{-\theta\lambda_2 t})+\theta(1-e^{-\theta\lambda_2 t})]$$
$$+[(e^{-(\lambda_2-\lambda_1)t}-\tfrac{\lambda_2}{\lambda_1})+(1+\tfrac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t})](1-e^{-(\lambda_2-\lambda_1)t})\Big]$$
$$\times\Big[[(e^{-(\lambda_2-\lambda_1)t}-\tfrac{\lambda_2}{\lambda_1})+(1+\tfrac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t})](e^{-(\lambda_2-\lambda_1)t}-\tfrac{\lambda_2}{\lambda_1})\Big]^{-1}$$

is non-negative. Simplifying the numerator yields

$$\Big[(e^{-(\lambda_2-\lambda_1)t}-\tfrac{\lambda_2}{\lambda_1})[\tfrac{\lambda_1}{\lambda_2}(1-e^{-\theta\lambda_2 t})+\theta(1-e^{-\theta\lambda_2 t})]$$
$$+(1+\tfrac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t})(1-e^{-(\lambda_2-\lambda_1)t})\Big]$$
$$\times\Big[[(e^{-(\lambda_2-\lambda_1)t}-\tfrac{\lambda_2}{\lambda_1})+(1+\tfrac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t})](e^{-(\lambda_2-\lambda_1)t}-\tfrac{\lambda_2}{\lambda_1})\Big]^{-1}.$$

At this time it becomes beneficial to break the proof up into two sub-cases: Case I in which $\lambda_1 > \lambda_2$, and Case II in which $\lambda_2 > \lambda_1$. Considering Case I, $\lambda_1 > \lambda_2$, the previous expression is of the form

$$\frac{(\text{pos})\,[(\text{pos})+(\text{pos})]+(\text{pos})\,(\text{neg})}{[(\text{pos})+(\text{pos})]\,(\text{pos})} \geq 0,$$

when only the sign of the bracketed quantities is considered. Thus to prove the hypothesis that $r_\theta(t) - r_0(t) \geq 0$ for $\lambda_1 > \lambda_2$, all that has to be shown is that the numerator is positive, as the denominator is always positive.

Proving the numerator,

$$\left(e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\right)\left(1-e^{-\theta\lambda_2 t}\right)\left(\frac{\lambda_1}{\lambda_2}+\theta\right) + \left(1-e^{-(\lambda_2-\lambda_1)t}\right)\left(1+\frac{\lambda_2}{\lambda_1}\theta - e^{-\theta\lambda_2 t}\right),$$

is positive is equivalent to showing that the inequality,

$$\left(1-e^{-\theta\lambda_2 t}\right)\left[\left(e^{-(\lambda_2-\lambda_1)t}\right)\left(\frac{\lambda_1}{\lambda_2}+\theta-1\right)+1-\frac{\lambda_2}{\lambda_1}\left(\frac{\lambda_1}{\lambda_2}+\theta\right)\right]$$

$$+\frac{\lambda_2}{\lambda_1}\theta\left(1-e^{-(\lambda_2-\lambda_1)t}\right) \geq 0,$$

holds. Multiplying by $\frac{\lambda_1}{\lambda_2}$ and then rearranging terms gives

$$e^{-(\lambda_2-\lambda_1)t}\left[\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_1}{\lambda_2}+\theta-1\right)\left(1-e^{-\theta\lambda_2 t}\right)-\theta\right]+\theta e^{-\theta\lambda_2 t} \geq 0 .$$

This can be multiplied through by $e^{\theta\lambda_2 t}$ to yield

$$e^{-(\lambda_2-\lambda_1-\theta\lambda_2)t}\left[\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_1}{\lambda_2}+\theta-1\right)\left(1-e^{-\theta\lambda_2 t}\right)-\theta\right]+\theta \geq 0.$$

This inequality becomes

$$e^{-(\lambda_2-\lambda_1-\theta\lambda_2)t}\left[\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_1}{\lambda_2}+\theta-1\right)\left(1-e^{-\theta\lambda_2 t}\right)\right]+\theta\left(1-e^{-(\lambda_2-\lambda_1-\theta\lambda_2)t}\right) \geq 0$$

when it is rearranged, and this in turn yields

$$\left[\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_1}{\lambda_2}+\theta-1\right)\left(1-e^{-\theta\lambda_2 t}\right)\right]+\theta\left(e^{-(\lambda_1-\lambda_2+\theta\lambda_2)t}-1\right) \geq 0$$

when multiplied by $e^{(\lambda_2-\lambda_1-\theta\lambda_2)t}$.

Denote the left hand side of the inequality by $g(t)$. This function of t is equal to zero when it is evaluated for $t = 0$. If the derivative of $g(t)$ can be shown to be greater than or equal to zero for all $t \geq 0$, then it follows that the function $g(t)$ is itself always non-negative.

The derivative of $g(t)$, which is to be proved non-negative, is

$$\theta\lambda_1\left(\frac{\lambda_1}{\lambda_2}+\theta-1\right)\left(e^{-\theta\lambda_2 t}\right)-\theta(\lambda_1-\lambda_2+\theta\lambda_2)e^{-(\lambda_1-\lambda_2+\theta\lambda_2)t} .$$

Proving the derivative is non-negative is equivalent to showing that

$$\theta \frac{\lambda_1}{\lambda_2} (\lambda_1+\theta\lambda_2-\lambda_2)e^{-\theta\lambda_2 t} \geq \theta(\lambda_1-\lambda_2+\theta\lambda_2)e^{-(\lambda_1-\lambda_2+\theta\lambda_2)t}$$

which simplifies to $\quad \frac{\lambda_1}{\lambda_2} \geq e^{-(\lambda_1-\lambda_2)t}$

when both sides are multiplied by $\quad \dfrac{e^{\theta\lambda_2 t}}{\theta(\lambda_1-\lambda_2+\theta\lambda_2)}$.

It is immediate, as $\lambda_1 > \lambda_2$ by assumption in Case I, that the above inequality holds. Therefore, for all cases where $\lambda_1 > \lambda_2$, $r_\theta(t)$ is greater than or equal to $r_0(t)$.

If Case II, $\lambda_1 < \lambda_2$, can now be shown, the proof of $r_0(t) \leq r_\theta(t)$ will be complete for all λ_1 and λ_2. From Expression 3.1, it is established that it is necessary to prove

$$\Bigg[\Big(e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\Big)\Big[\Big(e^{-(\lambda_2-\lambda_1)t} -1\Big) + \frac{\lambda_1}{\lambda_2}\Big(1-e^{-\theta\lambda_2 t}\Big) +\theta\Big(1-e^{-\theta\lambda_2 t}\Big)\Big]$$

$$+ \Big[\Big(e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\Big) + \Big(1+ \frac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t}\Big)\Big]\Big(1-e^{-(\lambda_2-\lambda_1)t}\Big)\Bigg]$$

$$\times \Bigg[\Big[\Big(e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\Big) + \Big(1+ \frac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t}\Big)\Big]\Big(e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\Big)\Bigg]^{-1} \geq 0.$$

Multiplying both sides of the inequality by $\Big(e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\Big)$, which is negative for $\lambda_1 < \lambda_2$, the inequality becomes

$$\frac{\Big(e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\Big)\Big(\frac{\lambda_1}{\lambda_2} +\theta\Big)\Big(1-e^{-\theta\lambda_2 t}\Big)+\Big(1+ \frac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t}\Big)\Big(1-e^{-(\lambda_2-\lambda_1)t}\Big)}{\Big[\Big(- \frac{\lambda_2}{\lambda_1} +e^{-(\lambda_2-\lambda_1)t}\Big) + \Big(1+ \frac{\lambda_2}{\lambda_1}\theta-e^{-\theta\lambda_2 t}\Big)\Big]} \leq 0.$$

Rearranging terms yields

$$\frac{\Big(1-e^{-\theta\lambda_2 t}\Big)\Big(\frac{\lambda_1}{\lambda_2} +\theta-1\Big)e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}\theta\Big(e^{-(\lambda_2-\lambda_1)t} - e^{-\theta\lambda_2 t}\Big)}{\Big(1-e^{-\theta\lambda_2 t}\Big) + e^{-(\lambda_2-\lambda_1)t} - \frac{\lambda_2}{\lambda_1}(1-\theta)} \leq 0,$$

which can be multiplied by $\dfrac{e^{(\lambda_2-\lambda_1)t}}{e^{(\lambda_2-\lambda_1)t}}$ to get

$$\frac{(1-e^{-\theta\lambda_2 t})(\frac{1}{\lambda_2})(\lambda_1-\lambda_2(1-\theta))-\frac{\lambda_2}{\lambda_1}\theta(1-e^{-(\lambda_1-\lambda_2(1-\theta))t})}{(1-e^{-(\lambda_1-\lambda_2(1-\theta))t})+(e^{-(\lambda_1-\lambda_2)t})(\frac{1}{\lambda_1})(\lambda_1-\lambda_2(1-\theta))} \leq 0.$$

Remember that $\lambda_1 < \lambda_2$ in this case. Now, if $\lambda_1 - \lambda_2(1 - \theta) > 0$, the form of the equation with regard to sign becomes [(pos) - (pos)]/ [(pos) + (pos)] ≤ 0. This implies the numerator has to be negative when $\lambda_1 - \lambda_2(1 - \theta) > 0$. On the other hand, if $\lambda_1 - \lambda_2(1 - \theta) < 0$, the form of the equation with regard to sign becomes [(neg) - (neg)]/ [(neg) + (neg)] $\leq$ 0. When $\lambda_1 - \lambda_2(1 - \theta) < 0$, the numerator has to be positive.

Denote the numerator by $g(t)$. This function is

$$(1-e^{-\theta\lambda_2 t})(\frac{1}{\lambda_2})(\lambda_1-\lambda_2(1-\theta))-\frac{\lambda_2}{\lambda_1}\theta(1-e^{-(\lambda_1-\lambda_2(1-\theta))t}).$$

Once again, since $g(t)$ evaluated at $t = 0$ is zero, it is sufficient to show that the derivative of $g(t)$ is always positive in order to prove the numerator is positive. Likewise, if the derivative of $g(t)$ is always negative for certain values of λ_1 and λ_2, it implies that the numerator is negative.

The derivative of $g(t)$ is

$$\theta\lambda_2 e^{-\theta\lambda_2 t}(\frac{1}{\lambda_2})(\lambda_1-\lambda_2(1-\theta)) - \frac{\lambda_2}{\lambda_1}\theta(\lambda_1-\lambda_2(1-\theta))e^{-(\lambda_1-\lambda_2(1-\theta))t},$$

which can be written as

$$\theta(\lambda_1-\lambda_2(1-\theta))\left[e^{-\theta\lambda_2 t}-\frac{\lambda_2}{\lambda_1}e^{-(\lambda_1-\lambda_2(1-\theta))t}\right].$$

In this form it can be seen that if $\lambda_1 - \lambda_2(1 - \theta) < 0$, then the derivative of $g(t)$ is positive, which implies that the numerator is positive.

If $\lambda_1 - \lambda_2(1 - \theta) > 0$, then the derivative of $g(t)$ is negative, which implies that the numerator is negative. Therefore, for $\lambda_1 < \lambda_2$, $r_\theta(t)$ is greater than or equal to $r_0(t)$.

For the special case where $\lambda_1 = \lambda_2$, $r_0(t) \leq r_\theta(t)$ is proved in Scroggins (Ref. 1). Thus, the proposition that the cold standby system hazard rate is a lower bound on the hazard rates for the hot and warm standby systems holds, as it has been shown that $r_0(t) \leq r_\theta(t)$ for all $t \geq 0$, $0 \leq \theta \leq 1$, and $\lambda_1 + \lambda_2 = 1$.

IV. AN OPTIMAL POLICY FOR THE USE OF COMPONENTS IN A STANDBY SYSTEM

A. PROPOSITION OF AN OPTIMAL COMPONENT EMPLOYMENT POLICY

Another conjecture advanced by Scroggins (Ref. 1) deals with the interchangeability of components. It is believed that the hazard rate of a system which uses the component with the larger failure rate as the primary component is always less than or equal to the hazard rate of the system in which the more reliable component is used first. This implies that by using the component with the larger failure rate as the primary component, and the other component as a spare, that the best overall system reliability can be achieved.

Figures 4.1 and 4.2 from Scroggins (Ref. 1) give examples of what is expected of warm standby systems in general. In the case of a hot standby system ($\theta = 1$) or a cold standby system ($\theta = 0$) it is immaterial as to which component is the primary component in the system and which component is the spare, as the hazard rates are equal for both possible employments of the components. This in turn implies the survival functions are equal for both the hot and cold standby systems no matter how the spare and primary components are employed. This fact can be readily checked by substituting either 0 or 1 in for θ in the hazard rate function in Equation 2.3 for any values of λ_1 and λ_2, and then comparing this equation to the same equation after reversing the values λ_1 and λ_2.

The above conjecture can be formalized by the following proposition, where $r_{\lambda_1, \lambda_2}(t)$ denotes the hazard rate of the warm standby system

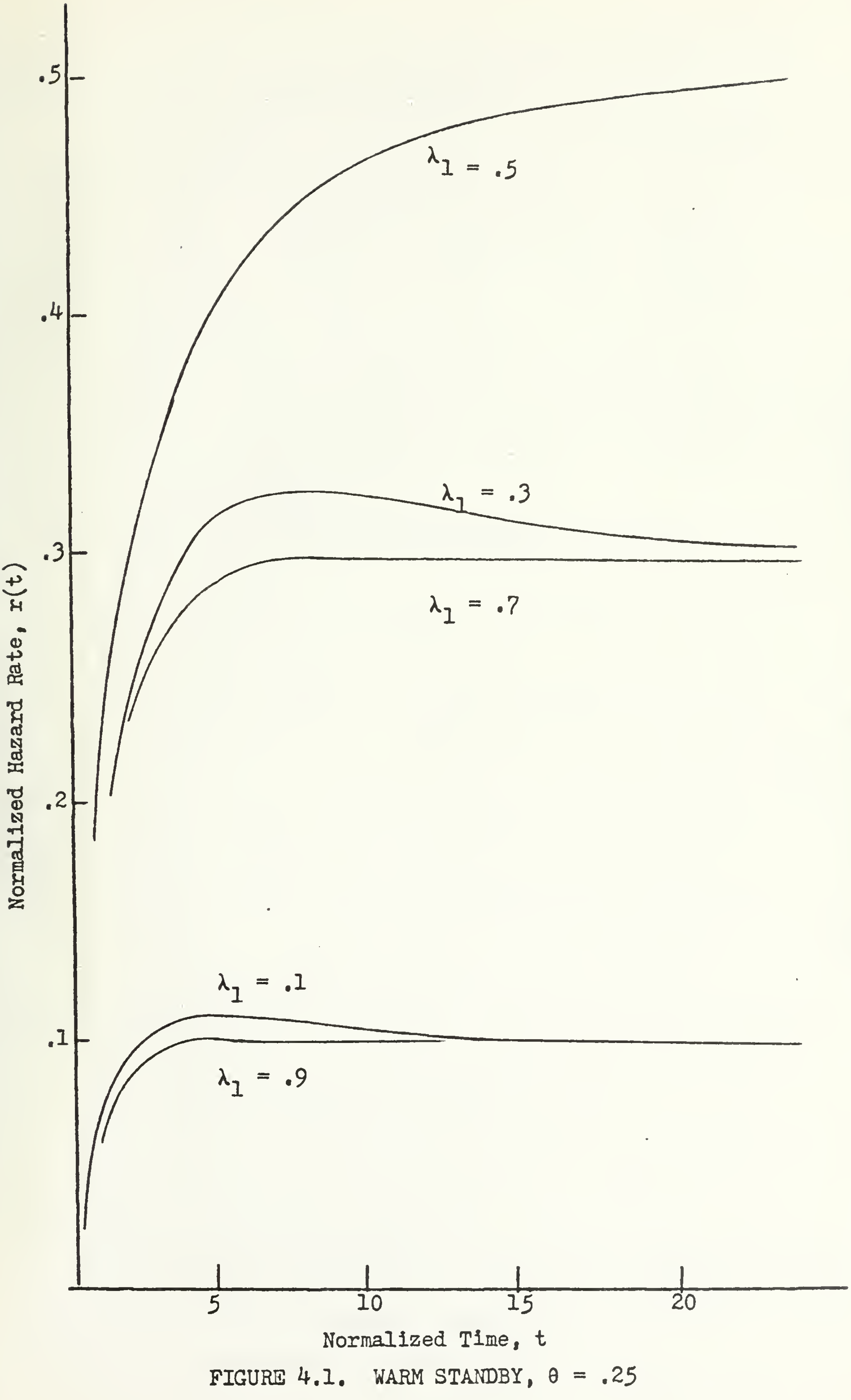

FIGURE 4.1. WARM STANDBY, $\theta = .25$

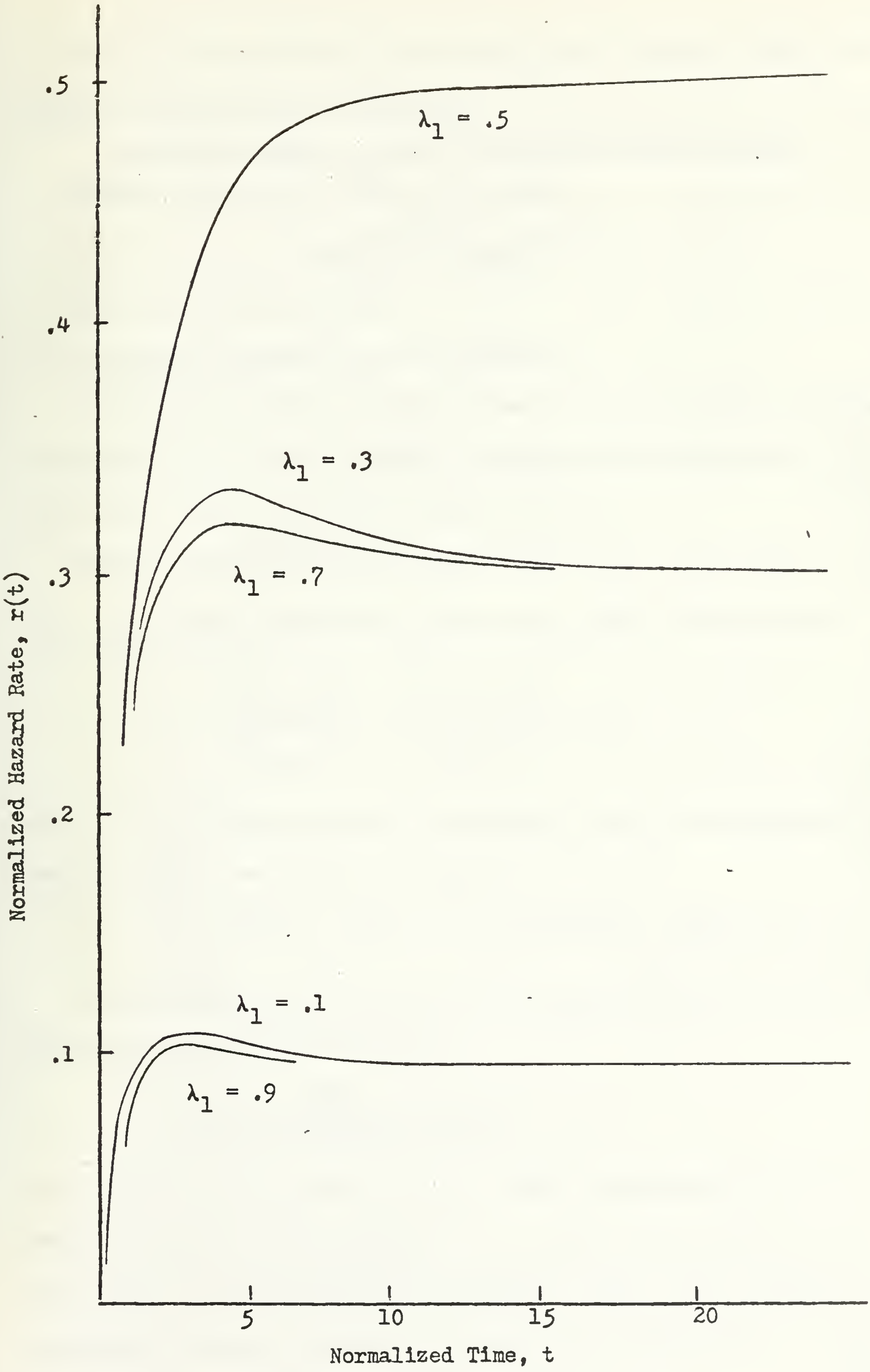

FIGURE 4.2. WARM STANDBY, $\theta = .75$

in which λ_1 is the failure rate of the primary component and λ_2 is the failure rate of the spare component.

Given two components, with normalized constant failure rates λ_1 and λ_2 such that $\lambda_1 > \lambda_2$, then

$$r_{\lambda_1,\lambda_2}(t) \leq r_{\lambda_2,\lambda_1}(t) ,$$

where $t \geq 0$ and $0 \leq \theta \leq 1$.

B. SUPPORTING EVIDENCE FOR THE PROPOSITION

In this section the previous proposition is not proved. Evidence is presented that makes it appear plausible that the proposition is true, but further research is needed to prove it.

Using the definition of the hazard rate, $r_{\lambda_1, \lambda_2}(t) = f_{\lambda_1, \lambda_2}(t)/\bar{F}_{\lambda_1, \lambda_2}(t)$, the proposition in this section can be rewritten as

$$\frac{f_{\lambda_2,\lambda_1}(t)}{\bar{F}_{\lambda_2,\lambda_1}(t)} - \frac{f_{\lambda_1,\lambda_2}(t)}{\bar{F}_{\lambda_1,\lambda_2}(t)} \geq 0,$$

if $\lambda_1 > \lambda_2$. Because survival functions are always non-negative, this can be further transcribed to become

$$f_{\lambda_2,\lambda_1}(t)\bar{F}_{\lambda_1,\lambda_2}(t) - f_{\lambda_1,\lambda_2}(t)\,\bar{F}_{\lambda_2,\lambda_1}(t) \geq 0.$$

With the appropriate substitutions using Equations 2.1 and 2.2 the above inequality assumes the form

$$\frac{Q(\theta)}{[\lambda_1-(1-\theta)\lambda_2]\,[\lambda_2-(1-\theta)\lambda_1]} \geq 0,$$

where $t \geq 0$, $\lambda_1 > \lambda_2$, and $0 \leq \theta \leq 1$. $Q(\theta)$, the numerator of the term on the left, has to be negative if $(1 - \theta)\,\lambda_1 > \lambda_2$ and positive if $(1 - \theta)\,\lambda_1 < \lambda_2$ for the above inequality to hold. This leads to the conclusion that as a function of θ, $Q(\theta)$ is negative from 0 to $\dfrac{\lambda_1-\lambda_2}{\lambda_1}$,

and then becomes positive from $\frac{\lambda_1-\lambda_2}{\lambda_1}$ to 1. Figure 4.3 indicates a postulated shape for the function $Q(\theta)$ that would verify the proposition.

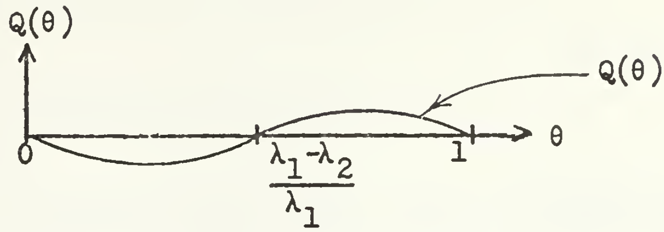

FIGURE 4.3. CONJECTURED SHAPE OF $Q(\theta)$

$Q(\theta)$ can be written as

$$\Big[\big[\lambda_2(\lambda_2-(1-\theta)\lambda_1)e^{-\lambda_2 t}+\lambda_1\lambda_2 e^{-\lambda_1 t}-\lambda_2(\lambda_2+\theta\lambda_1)e^{-(\lambda_2+\theta\lambda_1)t}\big]$$
$$\times\ \big[(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-(\lambda_1+\theta\lambda_2)t}\big]\Big]$$
$$-\ \Big\{\big[\lambda_1(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1(\lambda_1+\theta\lambda_2)e^{-(\lambda_1+\theta\lambda_2)t}\big]$$
$$\times\ \big[(\lambda_2-(1-\theta)\lambda_1)e^{-\lambda_2 t}+\lambda_2 e^{-\lambda_1 t}-\lambda_2 e^{-(\lambda_2+\theta\lambda_1)t}\big]\Big\}.$$

This simplifies to

$$e^{-t}\big[(\lambda_2-\lambda_1)\big[(\lambda_2-(1-\theta)\lambda_1)(\lambda_1-(1-\theta)\lambda_2)-\lambda_1\lambda_2\big]\big]$$
$$+\ e^{-(1+\theta)t}\big[\lambda_1\lambda_2(\lambda_2-\lambda_1)(1-\theta)\big]+e^{-(1+\lambda_1\theta)t}\big[\lambda_2(\lambda_1-(1-\theta)\lambda_2)(\lambda_1(1-\theta)-\lambda_2)\big]$$
$$+\ e^{-(1+\lambda_2\theta)t}\big[\lambda_1(\lambda_2-(1-\theta)\lambda_1)(\lambda_1-(1-\theta)\lambda_2)\big]$$
$$+\ e^{-(2\lambda_1+\theta\lambda_2)t}\big[\lambda_1\lambda_2^2\theta\big]-e^{-(2\lambda_2+\theta\lambda_1)t}\big[\lambda_1^2\lambda_2\theta\big].$$

Evaluating $Q(\theta)$ for $\theta = 0$, $\theta = 1$, and $\theta = \frac{\lambda_1-\lambda_2}{\lambda_1}$ yields zero in all cases. This is the first evidence that the postulated shape for the function $Q(\theta)$ may be correct.

Now turning to the derivative of $Q(\theta)$, it is seen that

$$\frac{dQ(\theta)}{d\theta} = e^{-t}[(\lambda_2-\lambda_1)[\lambda_1(\lambda_1-(1-\theta)\lambda_2)+\lambda_2(\lambda_2-(1-\theta)\lambda_1)]]$$

$$+ e^{-(1+\theta)t}[\lambda_1\lambda_2(\lambda_1-\lambda_2)(1+t(1-\theta))]$$

$$+ e^{-(1+\lambda_1\theta)t}[(\lambda_1(1-\theta)-\lambda_2)[\lambda_2^2-\lambda_1\lambda_2 t(\lambda_1-(1-\theta)\lambda_2)] -\lambda_1\lambda_2(\lambda_1-(1-\theta)\lambda_2)]$$

$$+ e^{-(1+\lambda_2\theta)t}[(\lambda_1-\lambda_2(1-\theta))[\lambda_1^2-\lambda_1\lambda_2 t(\lambda_2-(1-\theta)\lambda_1)] +\lambda_1\lambda_2(\lambda_2-(1-\theta)\lambda_1)]$$

$$+ e^{-(2\lambda_1+\theta\lambda_2)t}[\lambda_1\lambda_2^2 -\lambda_1\lambda_2^3 t\theta]$$

$$+e^{-(2\lambda_2+\theta\lambda_1)t}[-\lambda_1^2\lambda_2+\lambda_1^3\lambda_2 t\theta] .$$

Evaluated at $\theta = 0$, the derivative becomes

$$e^{-t}[(\lambda_1-\lambda_2)[-(\lambda_1-\lambda_2)^2+\lambda_1\lambda_2(1+t)+\lambda_2^2+\lambda_1^2-2\lambda_1\lambda_2]]$$

$$+ e^{-2\lambda_1 t}[\lambda_1\lambda_2^2] - e^{-2\lambda_2 t}[\lambda_1^2\lambda_2],$$

which equals

$$e^{-t}[(\lambda_1-\lambda_2)\lambda_1\lambda_2(1+t)] +e^{-2\lambda_1 t}[\lambda_1\lambda_2^2] -e^{-2\lambda_2 t}[\lambda_1^2\lambda_2].$$

If the proposition under investigation in this section is correct, the derivative of $Q(\theta)$ evaluated at $\theta = 0$ is less than or equal to zero. This is equivalent to the inequality

$$e^{-t}[\lambda_1\lambda_2(\lambda_1-\lambda_2)(1+t)] +e^{-2\lambda_1 t}[\lambda_1\lambda_2^2] -e^{-2\lambda_2 t}[\lambda_1^2\lambda_2]\leq 0.$$

Dividing through by $\lambda_1\lambda_2 e^{-t}$ and rearranging gives

$$\lambda_1(1+t-e^{-(\lambda_2-\lambda_1)t}) - \lambda_2(1+t-e^{-(\lambda_1-\lambda_2)t}) \leq 0.$$

Denote the left side of the above inequality by h(t). For any h(t), if h(0) = 0 and dh(t)/dt $\leq$ 0, then h(t) $\leq$ 0. This is the same argument used in the proof in the previous section.

h(t) does, in fact, equal zero for t = 0. The derivative of h(t) is

$$\lambda_1+\lambda_1(\lambda_2-\lambda_1)e^{-(\lambda_2-\lambda_1)t}-\lambda_2-\lambda_2(\lambda_1-\lambda_2)e^{-(\lambda_1-\lambda_2)t} .$$

This can in turn be denoted g(t). g(t) also equals zero for t = 0. The derivative of g(t) is

$$(\lambda_1-\lambda_2)[\lambda_1(\lambda_2-\lambda_1)e^{-(\lambda_2-\lambda_1)t}+\lambda_2(\lambda_1-\lambda_2)e^{-(\lambda_1-\lambda_2)t}].$$

Proving the derivative of g(t) is less than or equal to zero is equivalent to the inequality

$$(\lambda_1-\lambda_2)^2[\lambda_2 e^{-(\lambda_1-\lambda_2)t}-\lambda_1 e^{-(\lambda_2-\lambda_1)t}] \leq 0.$$

Since $\lambda_1 > \lambda_2$, it is immediate that the inequality does hold. This implies that g(t) is negative, which implies that h(t) is negative, which in turn implies that the derivative of Q(θ) is negative for θ = 0.

Evaluated at θ = 1, the derivative of Q(θ) is

$$\begin{aligned}
&e^{-t}[(\lambda_2-\lambda_1)(\lambda_1^2+\lambda_2^2)] \quad +e^{-2t}[\lambda_1\lambda_2(\lambda_1-\lambda_2)] \\
&- e^{-(1+\lambda_1)t}[\lambda_2(\lambda_2^2-\lambda_1^2\lambda_2 t)+\lambda_1^2\lambda_2] \\
&+ e^{-(1+\lambda_2)t}[\lambda_1(\lambda_1^2-\lambda_1\lambda_2^2 t)+\lambda_1\lambda_2^2] \\
&+ e^{-(2\lambda_1+\lambda_2)t}[\lambda_1\lambda_2^2-\lambda_1\lambda_2^3 t]-e^{-(2\lambda_2+\lambda_1)t}[\lambda_1^2\lambda_2+\lambda_1^3\lambda_2 t] .
\end{aligned}$$

Once again the derivative of Q(θ) is thought to be negative for θ = 1. If this is true, then the inequality,

$$e^{-t}\left[-\lambda_1^3+\lambda_2^3-\lambda_1\lambda_2^2+\lambda_1^2\lambda_2\right]+e^{-2t}\left[\lambda_1^2\lambda_2-\lambda_1\lambda_2^2\right]$$

$$+\ e^{-(1+\lambda_1)t}\left[\lambda_1\lambda_2^2-\lambda_1\lambda_2^3t-\lambda_2^3+\lambda_1^2\lambda_2^2t-\lambda_1^2\lambda_2\right]$$

$$+\ e^{-(1+\lambda_2)t}\left[-\lambda_1^2\lambda_2+\lambda_1^3\lambda_2t+\lambda_1^3-\lambda_1^2\lambda_2^2t+\lambda_1\lambda_2^2\right]\leq 0,$$

must hold. Multiplying both sides of the inequality by e^t gives

$$\left[-\lambda_1^3+\lambda_2^3-\lambda_1\lambda_2^2+\lambda_1^2\lambda_2\right]+e^{-t}\left[\lambda_1^2\lambda_2-\lambda_1\lambda_2^2\right]$$

$$+\ e^{-\lambda_1t}\left[\lambda_1\lambda_2^2-\lambda_1\lambda_2^3t-\lambda_2^3+\lambda_1^2\lambda_2^2t-\lambda_1^2\lambda_2\right]$$

$$+\ e^{-\lambda_2t}\left[-\lambda_1^2\lambda_2+\lambda_1^3\lambda_2t+\lambda_1^3-\lambda_1^2\lambda_2^2t+\lambda_1\lambda_2^2\right]\leq 0.$$

Again denote the left hand side of this inequality by $h(t)$. As before, $h(t)$ equals zero for $t = 0$. The derivative of $h(t)$ is negative if

$$\left[\lambda_1\lambda_2^2-\lambda_1^2\lambda_2\right]e^{-t}+\left[\lambda_1^2\lambda_2^3t-\lambda_1^3\lambda_2^2t+\lambda_1^3\lambda_2\right]e^{-\lambda_1t}$$

$$+\left[\lambda_1^2\lambda_2^3t-\lambda_1^3\lambda_2^2t-\lambda_1\lambda_2^3\right]e^{-\lambda_2t}\leq 0.$$

Dividing by $\lambda_1\lambda_2e^{-t}$ gives

$$(\lambda_2-\lambda_1)+\left[\lambda_1\lambda_2^2t-\lambda_1^2\lambda_2t+\lambda_1^2\right]e^{\lambda_2t}+\left[\lambda_1\lambda_2^2t-\lambda_1^2\lambda_2t-\lambda_2^2\right]e^{\lambda_1t}\leq 0.$$

Denoting the left side of this inequality by $g(t)$, it is possible to show that $g(t)$ equals zero for $t = 0$. The derivative of $g(t)$ is

$$e^{\lambda_2t}\left[\lambda_1\lambda_2^3t-\lambda_1^2\lambda_2^2t+\lambda_1\lambda_2^2\right]+e^{\lambda_1t}\left[-\lambda_1^3\lambda_2t+\lambda_1^2\lambda_2^2t-\lambda_1^2\lambda_2\right].$$

Factoring out the term $\lambda_1\lambda_2$, the derivative can be written in the form

$$(\lambda_1\lambda_2)\left[\lambda_2e^{\lambda_2t}(1+\lambda_2t-\lambda_1t)-\lambda_1e^{\lambda_1t}(1+\lambda_1t-\lambda_2t)\right]$$

which is always negative for all $t\geq 0$, as $\lambda_1>\lambda_2$.

The derivative of g(t) being negative, plus the fact that g(t) is zero when t = 0, implies that the derivative of h(t) is also negative. When the fact that h(t) is also zero when t = 0 is added to this information, it is proved that the derivative of $Q(\theta)$ is negative when $\theta = 1$.

In the preceeding paragraphs about the derivative of $Q(\theta)$, the term negative means non-positive rather than strictly negative. It is obvious that as a function of θ, the derivative of $Q(\theta)$ is zero for all $0 \leq \theta \leq 1$ if t is allowed to equal zero. If the special case where t = 0 is overlooked, then the derivatives of $Q(\theta)$ for $\theta = 0$ and $0 = 1$ are strictly negative.

The information known to be true about the function $Q(\theta)$ is displayed in Figure 4.4.

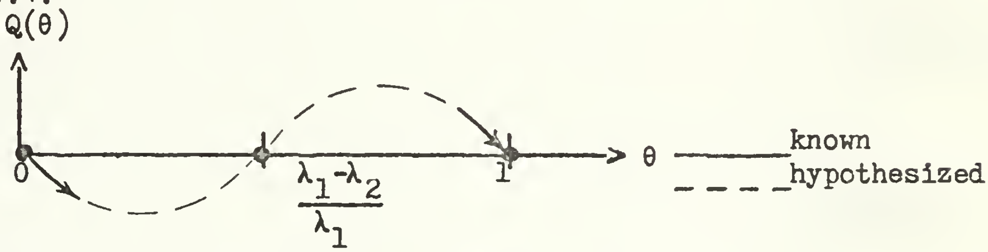

FIGURE 4.4. KNOWN FACTS ABOUT Q(θ)

The function's value has been proven to be zero for $\theta = 0$, $\theta = 1$, and $\theta = \frac{\lambda_1-\lambda_2}{\lambda_1}$. Furthermore, the derivative is known to be negative at $\theta = 0$ and $\theta = 1$. This information gives a strong indication that the postulated figure sketched in Figure 4.3 is true, which would imply that the proposition in this section is also true.

An attempt has been made to show that $Q(\theta)$ is negative for $0 \leq \theta \leq \frac{\lambda_1-\lambda_2}{\lambda_1}$, and then positive for values of θ such that $\frac{\lambda_1-\lambda_2}{\lambda_1} \leq \theta \leq 1$.

Another attempt was made to prove that the second derivative of $Q(\theta)$ with respect to θ is positive for $0 \leq \theta \leq \frac{\lambda_1-\lambda_2}{\lambda_1}$ and negative for

$\frac{\lambda_1 - \lambda_2}{\lambda_1} \leq \theta \leq 1$. If either of these conjectures about the function of $Q(\theta)$ could be proved, they would imply that the proposition in this section is true for all $t \geq 0$ and $0 \leq \theta \leq 1$. Further study in this area is needed.

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BSTRACT

Two propositions concerning the comparative behavior of the hazard rate unctions for warm, cold, and hot standby systems are expressed and then xamined. The standby systems under study involve two components with contant failure rates. Emphasis is placed on establishing a lower bound on he hazard rate function for a standby system and on developing an optimal mployment policy for the use of components in a standby system.

A proof is offered to show that the hazard rate function for a cold tandby system is a lower bound on the hazard rate function of either a arm or hot standby system. Evidence is then given to support the exis-tence of an optimal employment policy for components in a standby system, ut a complete proof is not presented.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Warm Standby System						
Hazard Rate						
Exponential Distribution						
Lower Bound on Warm Standby System Hazard Rate						
Optimal use of Components in Warm Standby System						

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